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AN F-SPACE WITH TRIVIAL DUAL AND NON-TRIVIAL COMPACT ENDOMORPHISMS

BY

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ABSTRACT

We give an example of an F-space which has non-trivial compact endomorphisms, but does not have any non-trivial continuous linear functionals.

1. Introduction

The object of this paper is to give an example of an F-space (complete, metrizable linear topological space) which has non-trivial compact endomorphisms but does not have non-trivial continuous linear functionals. An additional curious property of this space is that its algebra of continuous endomorphisms is not transitive; that is, there exist non-zero vectors f and g in the space such that no continuous endomorphism takes f to g.

In the opposite direction, D. Pallaschke [6] and P. Turpin [9] have recently shown that certain F-spaces of measurable functions already known to have trivial duals, in particular the spaces $L^p([0,1]$ for $0<p<1$, have no non-trivial compact endomorphisms.

Our example is constructed from the classical Hardy space H^p of analytic functions $(0<p<1)$, and relies heavily on the existence of certain rather explicitly determined proper, closed, weakly dense subspaces recently discovered by P. L. Duren, B. W. Romberg, and A. L. Shields [2]. The necessary background material is outlined in the next section, after which the example is constructed.

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2. Preliminaries on H^p

A good reference for the material in this section is Duren's book [1], especially chapters 2 and 7. In what follows, Δ denotes the open unit disc in the complex plane. For $0 < p < \infty$ *the Hardy space* H^p is the collection of functions f analytic in Δ for which

$$\|f\|_p = \sup_{0 \leqq r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right\}^{1/p} < \infty.$$

H^p is a linear space over the complex numbers, and if $1 \leqq p < \infty$ then $\|\cdot\|_p$ is a norm which makes it into a Banach space. For $0 < p < 1$, the case of interest to us, the p-homogeneous functional $\|\cdot\|_p^p$ is subadditive and induces a translation-invariant metric

$$d(f,g) = \|f - g\|_p^p$$

on H^p which makes it into an F-space [1, p. 37, corollary 2].

When $0 < p < 1$ the functional $\|\cdot\|_p^p$ is not homogeneous, and the topology it induces on H^p is not locally convex [5]. We will nevertheless refer to $\|\cdot\|_p^p$ as the *norm* on H^p, and call the corresponding topology the *norm topology*. Note that the positive multiples of the unit ball

$$\{f \in H^p : \ \|f\|_p^p \leqq 1\}$$

form a local base for the norm topology; and therefore a subset B of H^p is (topologically) bounded if and only if it is *norm bounded*:

$$\sup\{\|f\|_p^p : \ f \in B\} < \infty.$$

PROPOSITION 2.1. *Every bounded subset of H^p is a normal family* $(0 < p < \infty)$.

PROOF. We have the estimate

$$|f(z)| \leqq 2^{1/p} \|f\|_p (1 - |z|)^{-1/p} \qquad (z \in \Delta)$$

for $f \in H^p$ [1, p. 36], so if B is a bounded subset of H^p, then the members of B are bounded uniformly on compact subsets of Δ. It follows that B is a normal family [7, theorem 14.6, p. 272].

Let κ denote the restriction to H^p of the topology of uniform convergence on compact subsets of Δ. It is well known that κ is locally convex and metrizable.

PROPOSITION 2.2. *The closed unit ball of H^p is κ-compact* $(0 < p < \infty)$.

PROOF. Since κ is metrizable it is enough to show that each sequence in the closed unit ball U of H^p has a subsequence κ-convergent to an element of U. Suppose (f_n) is a sequence in U. Since U is a normal family (Proposition 2.1) there is subsequence (f_{n_j}) and an analytic function f on Δ such that $f = \kappa - \lim f_{n_j}$. For $0 \leqq r < 1$ the sequence (f_{n_j}) converges to f uniformly on the circle $|z| = r$, hence

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt = \lim_{j\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |f_{n_j}(re^{it})|^p dt \leqq 1,$$

from which it follows that $\|f\|_p \leqq 1$. Thus $f \in U$ and the proof is complete.

One of the most important facts about H^p spaces is that for each f in H^p the *radial limit*

$$f^*(e^{it}) = \lim_{r\to 1-} f(re^{it})$$

exists for almost every real t [1, theorem 2.2, p. 17], and moreover

$$\|f\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{it})|^p dt \tag{2.1}$$

[1, theorem 2.6, p. 21]. A function q analytic in Δ is called an *inner function* if $|q| \leqq 1$ on Δ, and $q^*(e^{it}) = 1$ a.e. It follows from Eq. (2.1) that for q inner the multiplication map $f \to qf$ is an isometry on H^p, so its range qH^p is a closed subspace. If, moreover, q is *non-trivial* (i.e. $q \not\equiv 1$), then the Canonical Factorization Theorem [1, theorem 2.8, p. 24] shows that qH^p is a *proper* subspace of H^p. We can now state a result of Duren, Romberg, and Shields which provides the key to our example.

THEOREM 2.3. [2, theorem 13, p. 53]. *There exists a non-trivial inner function q such that qH^p is dense in weak topology of H^p for $0 < p < 1$.*

We remark that every non-locally convex F-space has a closed subspace that is not weakly closed. By [2, theorems 16 and 17, p. 59] an equivalent assertion is that the extension form of the Hahn-Banach theorem fails in every non-locally convex F-space; and Kalton [3, corollary 5.3] has recently proved this latter assertion. We do not know if every non-locally convex F-space must have a proper, closed, weakly dense subspace. According to Theorem 2.3, such subspaces exist in H^p $(0 < p < 1)$. They have also been found in certain other F-spaces of analytic functions, as well as in l^p $(0 < p < 1)$ [8].

3. The example

Suppose E and F are linear topological spaces. A linear transformation $T: E \to F$ is said to be *compact* (or *completely continuous*) if there is a neighborhood of 0 in E whose image under T is compact in F. It is easy to see that every compact linear transformation is continuous, and that the composition of a compact and a continuous linear transformation (in either order) is again compact.

The collection of continuous linear functionals on E is called the *dual* of E, denoted by E'. If $E' = \{0\}$ we say E has *trivial dual*. An *endomorphism* of E is a linear transformation of E into itself. We now state our main result.

THEOREM 3.1. *There is an F-space with trivial dual which has non-trivial compact endomorphisms.*

The proof will require some preliminaries. In particular we need a certain topology on H^p intermediate between κ (the topology of uniform convergence on compact subsets of Δ) and the norm topology. This topology, denoted by β, is *the strongest topology on H^p that agrees with κ on every norm bounded subset*. It is easy to see that such a topology exists: we simply declare a set to be β-open (respectively β-closed) if its intersection with every bounded set B is relatively κ-open (respectively κ-closed) in B. It is easy to see that these "β-open" sets actually satisfy the axioms for a topology. Since a subset of H^p is bounded if and only if it is norm bounded, if we wish to decide whether a set is β-open or β-closed, then we need only consider its intersection with every positive (or even positive integer) multiple of the closed unit ball.

Since β is stronger than the Hausdorff topology κ, it is itself Hausdorff. Since the closed unit ball of H^p is κ-compact (Proposition 2.1), it is also β-compact. Finally, β is weaker than the norm topology of H^p, since it is weaker on bounded sets. To summarize:

PROPOSITION 3.2. *β is a Hausdorff topology on H^p intermediate between κ and the norm topology. The closed unit ball of H^p is β-compact* $(0 < p < \infty)$.

It will be important for us to known that β is actually a vector topology — a fact we have not assumed in advance. We give a proof modelled after that of the Banach-Dieudonné theorem [4, sec. 2.2, pp. 211–212].

PROPOSITION 3.3. *β is a vector topology.*

PROOF. For brevity we will refer to a κ-closed κ-neighborhood of zero as an *admissible* neighborhood. We denote the closed unit ball of H^p by U. It is easy to check that the collection of all sets of the form

$$\bigcap_{n=1}^{\infty} (p_n U + V_n) \tag{3.1}$$

where (p_n) is a real sequence with $0 \leqq p_n \to \infty$, and (V_n) is a sequence of admissible neighborhoods, is a local base for a vector topology β' on H^p (in fact, it follows from the work of Wiweger [10, sec. 2.3, p. 52], that β' is the strongest vector topology on H^p agreeing with κ on bounded sets). We are going to show that $\beta' = \beta$.

To see that $\beta' \leqq \beta$, suppose B is a bounded subset of H^p, so $B \subseteq kU$ for some $k > 0$. Suppose N is a β'-neighborhood of zero of the form (3.1). Then whenever $p_n \geqq k$ we have

$$p_n U + V_n \supseteq kU \supseteq B,$$

so

$$B \cap N = B \cap \bigcap_{p_n \leqq k} (p_n U + V_n) = B \cap W$$

where

$$W = \bigcap_{p_n \leqq k} (p_n U + V_n)$$

is a κ-neighborhood of zero. Thus β' agrees with κ on bounded sets, so $\beta' \leqq \beta$.

In the other direction, suppose A is a β-open set containing the origin. We claim that there exists a sequence (V_k) of admisible neighborhoods such that for each integer $n \geqq 1$:

$$nU \cap \bigcap_{k=1}^{n} [(k-1)U + V_k] \subseteq A. \tag{3.2}$$

Then the β'-neighborhood of zero

$$\bigcap_{k=1}^{\infty} [(k-1)U + V_k]$$

will be contained in A, completing the proof.

We obtain the sequence (V_k) by induction. Since $\beta = \kappa$ on U we know there is an admissible neighborhood V_1 such that

$$U \cap V_1 \subseteq U \cap A \subseteq A,$$

and this is just inequality (3.2) for $n = 1$. So suppose $V_1, \cdots, V_n$ are admissible neighborhoods satisfying (3.2). We want to find V_{n+1} so that $V_1, \cdots, V_{n+1}$ also satisfy (3.2). Suppose we cannot; that is, suppose

$$(3.3) \qquad (n+1)U \cap \bigcap_{k=1}^{n} [(k-1)U + V_k] \cap (nU + V) \cap A^C \neq \phi$$

for each admissible neighborhood V (here A^c denotes the complement of A in H^p). Since A is β-open and $(n+1)U$ is β-compact, the set $(n+1)U \cap A^c$ is β-compact, hence κ-compact. It follows from the κ-compactness of U that $\alpha U + V$ is κ-closed for every admissible neighborhood V. Thus the left side of (3.3) is, for each admissible V, a non-void κ-closed subset of the κ-compact space $(n+1)U$. It follows easily from (3.3) that the family of all these left sides has the finite intersection property, and hence a common point: that is,

$$(3.4) \qquad (n+1)U \cap \bigcap_{k=1}^{n} [(k-1)U + V_n] \cap \bigcap_{V} (nU + V) \cap A^c \neq \phi,$$

where V ranges over all admissible neighborhoods. Now the κ-compactness of nU guarantees that $\cap_V (nU + V) = nU$ [4; theorem 5.2 (v), p. 35], so (3.4) reduces to

$$nU \cap \bigcap_{k=1}^{n} [(k-1)U + V_k] \cap A^c \neq \phi,$$

which contradicts (3.2). This completes the proof.

We note in passing that this proof uses only the fact that H^p is a Hausdorff locally bounded linear topological space, and κ is a Hausdorff vector topology for which each norm bounded set is relatively compact. That is, we have really proved:

PROPOSITION 3.3′. *Suppose E is a Hausdorff locally bounded linear topological space and κ is a Hausdorff vector topology on E for which each norm bounded subset of E is relatively compact. Then the strongest topology that agrees with κ on each norm bounded set is actually a vector topology.*

Finally we need to know that certain norm-closed subspaces of H^p are also β-closed.

PROPOSITION 3.4. *For every inner function q the subspace qH^p is β-closed in H^p.*

PROOF. Let U denote the closed unit ball of H^p. It is enough to show that $U \cap qH^p$ is κ-closed in H^p. So suppose (f_n) is a sequence in $U \cap qH^p$, $f \in H^p$, and $f = \kappa - \lim f_n$. Now there exists a sequence (h_n) in H^p such that $f_n = qh_n$ for each n; and since

$$\|h_n\|_p = \|f_n\|_p \leqq 1$$

we have $(h_n) \subseteq U$. Since U is κ-compact (Proposition 2.1) there is a subsequence (h_{n_j}) which is κ-convergent to some h in U. Consequently

$$qh = \kappa - \lim qh_{n_j} = \kappa - \lim f_{n_j} = f,$$

so $f \in U \cap qH^p$, and the proof is complete.

PROOF OF THEOREM 3.1. Fix $0 < p < 1$. By Theorem 2.3 we can choose a non-trivial inner function q such that the (proper) closed subspace qH^p is weakly dense. Let E_N denote the quotient linear topological space H^p/qH^p, where H^p has its norm topology. Since qH^p is norm closed in H^p, E_N is Hausdorff: in fact it is complete and metrizable. Since qH^p is weakly dense in H^p, the quotient space E_N has trivial dual [2, corollary 1, p. 53].

Let E_β denote the quotient linear topological space H^p/qH^p, *where H^p has the β-topology.* E_β is Hausdorff since qH^p is β-closed (Proposition 3.4). Now E_N and E_β are the same linear space H^p/qH^p, but with different topologies. It is not difficult to see that the topology of E_β is weaker than that of E_N, since the β-topology is weaker on H^p than the norm topology. In particular the identity map $j_{N,\beta}$: $E_N \to E_\beta$ is continuous, and E_β also has trivial dual.

Let U denote the closed unit ball of H^p and let π denote the quotient map taking H^p onto H^p/qH^p. Then π is continuous when viewed as a map of H^p in its norm topology onto E_N, and also when viewed as a map of H^p in the β-topology onto E_β. In addition, $V = \pi(U)$ is the closed unit ball of E_N, and it is a compact subset of E_β, since U is β-compact in H^p. Thus the identity map $j_{N,\beta}$; $E_N \to E_\beta$ takes the closed unit ball of E_N onto a compact subset of E_β, and is therefore a compact linear transformation.

Now any vector topology can be represented as the least upper bound of a family of pseudo-metric topologies [4, section 6, problem C, p. 51–52]. In particular there is a pseudo-metric d on E_β that induces a (not necessarily Hausdorff) vector topology weaker than β, yet different from the indiscrete topology. Let E_d denote E_β equipped with this new topology: then the identity map $j_{\beta,d}$: $E_\beta \to E_d$ is continuous. Let F denote the closure of $\{0\}$ in E_d. Then F is a proper, closed subspace of E_d: and it is not difficult to see that the quotient space E_d/F is a non-trivial, *metrizable* linear topological space.

Since the quotient map ρ: $E_d \to E_d/F$ is continuous, so is its composition with $j_{\beta,d}$. Recall that the identity map $j_{N,\beta}: E_N \to E_\beta$ is compact; thus the composition $S = \rho \circ j_{\beta,d} \circ j_{N,\beta}$ is a non-trivial compact linear map taking E_N onto the (necessarily incomplete) linear metric space E_d/F. Let E_M be the completion of E_d/F. Then E_M is an F-space, and S can be regarded as a non-trivial compact linear transformation from E_N into E_M. Let $E = E_N \oplus E_M$ and define T: $E \to E$ by

$$T(x, y) = (0, Sx) \qquad (x \in E_N,\ y \in E_M).$$

Then E is an F-space since E_N and E_M are, and T is a non-trivial compact endomorphism of E.

It remains to show that E has non-trivial dual. We have already observed that E_N has trivial dual, as does E_β. Since E_d is just E_β in a weaker topology, it too has trivial dual, as does its quotient E_d/F. Thus E_M, which is the completion of E_d/F also has trivial dual, hence so does $E = E_N \oplus E_M$. This completes the proof.

The space E that we have constructed has a further curious property. A linear topological space is said to be *transitive* if for each pair f, g of non-zero vectors there is a continuous endomorphism of the space which takes f to g. For example, it is easy to see that any linear topological space whose dual separates points is transitive; while the direct sum of a space with trivial dual and its scalar field is not transitive. Pełczyński has observed that a transitive linear topological space with non-trivial compact endomorphisms must also have non-trivial dual. This result is stated and proved in [6, theorem 1.2, p. 125] for real scalars. The essential feature of the proof is an application of the Riesz theory of compact operators, which holds as well for complex scalars [4, chapter 5, problems A and B, pp. 206–207). So Pełczyński's result and its proof as given in [6] hold in the complex case, and we obtain from it and Theorem 3.1 the following:

COROLLARY. *There exists a non-transitive F-space with trivial dual.*

We note in closing that the space E constructed in Theorem 3.1 can be chosen to be locally bounded. To see this, let us fix $0 < p < 1$ and revert to the notation used in the proof of Theorem 3.1. Since the topology β on H^p has a local base of absolutely p-convex sets, so does the quotient space E_β (a subset S of a real or complex linear space is *absolutely p-convex* if $ax + by \in S$ whenever $x, y \in S$ and $|a|^p + |b| \leqq 1$). The "Minkowski functional" (cf. [4, p. 15]) of such a neighborhood is subadditive and absolutely p-homogeneous, and one of these functionals can be used to induce the pseudo-metric d. It follows quickly that the metric on E_M is also induced by a subadditive, p-homogeneous functional, hence E_M is locally bounded. Now E_N is locally bounded by the definition of the quotient metric, hence so is the direct sum $E = E_M \oplus E_N$, and our assertion is proved.

Added December 2, 1974. After this paper was submitted P. Turpin pointed out to us that Proposition 3.3′ is a special case of a result of L. Waelbroeck (*Topological Vector Spaces and Algebras*, Springer Lecture Notes in Mathematics, No 230, Proposition 6.2, p. 48).

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